On the Product of Matrix Exponentials

Shmuel Friedland
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60680

and

Wasin So*
Institute for Mathematics and Its Applications
University of Minnesota
Minneapolis, Minnesota

Submitted by Hans Schneider

ABSTRACT

We study the family of positive definite Hermitian matrices of the form $(e^{tB/2}e^{tA}e^{tB/2})^{1/t}$ for $t > 0$, where A and B are Hermitian. In particular, we show that the above matrix family converges to a limit when $t \to \infty$.

1. INTRODUCTION

Let M_n be the algebra of $n \times n$ complex-valued matrices. For $F \in M_n$ denote by

$$C_k(F) \in M_{\binom{n}{k}}$$

the kth compound of F. That is, the entries of $C_k(F)$ are all $k \times k$ minors of F. The row and the column of a specific $k \times k$ minor of F in $C_k(F)$ are determined by the lexicographical order on the corresponding k rows and columns which determine this minor. As usual, let $\rho(F)$ be the spectral

*Current address: Division of Math and Information, Sam Houston State University, Huntsville, TX 77341.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

radius of F. Let $H_n \subset M_n$ be the real subspace of Hermitian matrices. Note that

$$A \in H_n \quad \text{implies} \quad C_k(A) \in H_{\binom{n}{k}}.$$

Assume that $A \in H_n$. Let $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ be the eigenvalues of A arranged in decreasing order. Set $\operatorname{tr}_i A = \sum_{j=1}^{i} \alpha_j$ to be the ith partial trace of a Hermitian matrix A. Thus, $\operatorname{tr} A = \operatorname{tr}_n A$. For $2 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant \binom{n}{k}$ let $\operatorname{tr}_i^{(k)} A = \operatorname{tr}_i C_k(A)$. In particular, we let $\operatorname{tr}^{(k)} A = \operatorname{tr} C_k(A)$ and $\operatorname{tr}_i^{(1)} A = \operatorname{tr}_i A$. Finally note that $\operatorname{tr}^{(n)} A = \operatorname{tr}_1^{(n)} A = \det A$.

For $t > 0$, let $M(t)$ denote the positive definite Hermitian matrix $(e^{tB/2}e^{tA}e^{tB/2})^{1/t}$, where A and B are Hermitian matrices. In this paper we study the monotonicity of partial traces of $\operatorname{tr}_i^{(k)} M(t)$ as a function of t. We recall some known facts about $\operatorname{tr}_i^{(k)} M(t)$. First note that $\det M(t) = e^{\operatorname{tr}(A+B)}$ is always a constant. Second, in [1] it is proved that $\operatorname{tr}_1^{(k)} M(t)$ is an increasing function of t. In [3] it is shown that $\operatorname{tr} M(t)$ is an increasing function of t. Using the Lie product formula $\lim_{t\to 0} M(t) = e^{A+B}$, one obtains the Golden-Thompson inequality $\operatorname{tr} e^{A+B} \leqslant \operatorname{tr}(e^{A}e^{B})$. In [7] and [8] it is proved that $\operatorname{tr} e^{A+B} = \operatorname{tr}(e^{A}e^{B})$ iff $AB = BA$. According to a referee's remark the condition for the equality can be easily deduced from the original arguments of Golden [2].

We now summarize briefly our results. In the next section we show that the function $\operatorname{tr}_i^{(k)} M(t)$ is always increasing. In Section 3, we characterize the situations when these functions are not strictly increasing. Section 4 is devoted to proving the existence of $\lim_{t\to\infty} M(t)$. This result follows from a more general theorem.

2. ON THE FUNCTION $\operatorname{tr}_i^{(k)} M(t)$

In this paper we shall always assume that $A, B \in H_n$. As usual, let $B \geqslant A$ iff $B - A$ is nonnegative definite. Assume that $x^1, \ldots, x^n$ is an orthonormal system of eigenvectors of A corresponding to the eigenvalues $\alpha_1, \ldots, \alpha_n$:

$$Ax^i = \alpha_i x^i, \quad i = 1, \ldots, n, \qquad (x^j)^* x^i = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

Recall that for $f: \mathbf{R} \to \mathbf{R}$, $f(A) = \sum_{i=1}^{n} f(\alpha_i) x^i (x^i)^*$. In particular, if f is the characteristic function of $(0, \infty)$, i.e., $f(x) = 0$ for $x \leqslant 0$ and $f(x) = 1$ for $x > 0$, then $P_+(A) = f(A)$ is the projection of A on its positive spectrum. We now recall some well-known results on the wedge product and the compound matrices; see for example, [5]. First note that for any $F \in M_n$, in the wedge notation

$$C_k(F) = F \wedge \cdots \wedge F \in M_{\binom{n}{k}}.$$

Moreover, for $A \in H_n$, the $\binom{n}{k}$ eigenvalues and orthonormal eigenvectors of $C_k(A)$ are

$$C_k(A) x_{i_1} \wedge \cdots \wedge x_{i_k} = \alpha_{i_1} \cdots \alpha_{i_k} x_{i_1} \wedge \cdots \wedge x_{i_k},$$
$$1 \leqslant i_1 < \cdots < i_k \leqslant n.$$

Assume furthermore that $A \geqslant 0$. It then follows that $C_k(A) \geqslant 0$. In particular, the first (maximal) eigenvalue and the second eigenvalue of $C_k(A)$ are $\alpha_1 \cdots \alpha_k$ and $\alpha_1 \cdots \alpha_{k-1}\alpha_{k+1}$ respectively.

THEOREM 2.1. *Let* $A, B \in H_n$. *Set* $M(t) = (e^{tB/2}e^{tA}e^{tB/2})^{1/t} > 0$, $t > 0$. *Then for* $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant \binom{n}{k}$ *the function* $\operatorname{tr}_i^{(k)} M(t)$ *increases on* $(0, \infty)$.

Proof. We first prove that $\operatorname{tr}_1^{(k)} M(t)$ increases. As

$$e^{tA}e^{tB} = e^{-tB/2}\left(e^{tB/2}e^{tA}e^{tB/2}\right)e^{tB/2},$$

we deduce $M(t)^t$ and $e^{tA}e^{tB}$ have the same eigenvalues $\lambda_1(t) \geqslant \cdots \geqslant \lambda_n(t) > 0$. In particular

$$t \log \operatorname{tr}_1^{(k)} M(t) = \log \prod_{j=1}^{k} \lambda_j(t) = f_k(t). \tag{2.2}$$

According to [1, Corollary 10] the function $f_k(t)/t$ increases on $(0, \infty)$. Hence, $\operatorname{tr}_1^{(k)} M(t)$ is an increasing function. We now prove that $\operatorname{tr}_i M(t)$ increases on $(0, \infty)$. Let $\omega_1(t) \geqslant \cdots \geqslant \omega_n(t) > 0$ be the eigenvalues of $M(t)$. Hence, $\operatorname{tr}_1^{(k)} M(t) = \omega_1(t) \cdots \omega_k(t)$. As $\operatorname{tr}_1^{(k)} M(t)$ increases on $(0, \infty)$, it follows that

$$\begin{aligned} &\omega_1(t_1) \cdots \omega_k(t_1) \leqslant \omega_1(t_2) \cdots \omega_k(t_2), \qquad k = 1, \ldots, n-1, \\ &\omega_1(t_1) \cdots \omega_n(t_1) = \omega_1(t_2) \cdots \omega_n(t_2) = e^{\operatorname{tr}(A+B)}, \qquad 0 < t_1 < t_2. \end{aligned} \tag{2.3}$$

We now claim that the above inequalities for $k = 1, \ldots, i$, imply

$$\operatorname{tr}_i M(t_1) = \sum_{j=1}^{i} \omega_j(t_1) \leqslant \operatorname{tr}_i M(t_2) = \sum_{j=1}^{i} \omega_j(t_2).$$

Moreover, $\operatorname{tr}_i M(t_1) = \operatorname{tr}_i M(t_2)$ iff $\omega_k(t_1) = \omega_k(t_2)$, $k = 1, \dots, i$. Indeed, consider first the case $i = n$. It then follows that the vector $u = (\log \omega_1(t_1), \dots, \log \omega_n(t_1))$ is majorized by the vector $v = (\log \omega_1(t_2), \dots, \log \omega_n(t_2))$, i.e., $u \prec v$. (See the modern treatise of [6] on the subject of majorization.) As e^x is a convex function, the classical inequality due to Hardy, Littlewood, and Pólya yields the inequality $\operatorname{tr} M(t_1) \leqslant \operatorname{tr} M(t_2)$. As e^x is strictly convex, it follows that the equality holds iff $\omega_j(t_1) = \omega_j(t_2)$, $j = 1, \dots, n$. Let $i < n$. Set

$$a_i = \frac{\omega_1(t_1) \cdots \omega_i(t_1)}{\omega_1(t_2) \cdots \omega_{i-1}(t_2)} \leqslant \omega_i(t_2).$$

Hence,

$$(\log \omega_1(t_1), \dots, \log \omega_i(t_1)) \prec (\log \omega_1(t_2), \dots, \log \omega_{i-1}(t_2), \log a_i).$$

As e^x is a strictly convex function, we obtain that $\operatorname{tr}_i M(t_1) \leqslant \operatorname{tr}_{i-1} M(t_2) + a_i \leqslant \operatorname{tr}_i M(t_2)$. The equality $\operatorname{tr}_i M(t_1) = \operatorname{tr}_i M(t_2)$ holds iff $\omega_j(t_1) = \omega_j(t_2)$, $j = 1, \dots, i$. We now show that $\operatorname{tr}_i^{(k)} M(t)$ increases on $(0, \infty)$. First, recall that for any $F \in M_n$ we have the identity $C_k(e^{tF}) = e^{tD_k(F)}$. Here

$$D_k(F) \in M_{\binom{n}{k}}$$

is the additive kth compound of F. That is,

$$D_k(F) = \frac{d}{dt} C_k(I + tF)\Big|_{t=0}.$$

In particular,

$$A \in H_n \quad \text{implies} \quad D_k(A) \in H_{\binom{n}{k}}.$$

The multiplicativity property of the compound $[C_k(FG) = C_k(F)C_k(G)$, $F, G \in M_n]$ yields

$$C_k(e^{tB/2} e^{tA} e^{tB/2}) = e^{tD_k(B)/2} e^{tD_k(A)} e^{tD_k(B)/2}.$$

We thus can take the tth root of the above identity. Recall our remarks about the eigenvalues and the eigenvectors of the compound matrices of $C \in H_n$

and $f(C)$ to deduce the identity

$$C_k(M(t)) = M_k(t) = \left(e^{tD_k(B)/2}e^{tD_k(A)}e^{tD_k(B)/2}\right)^{1/t}. \qquad (2.4)$$

As $\operatorname{tr}_i^{(k)} M(t) = \operatorname{tr}_i M_k(t)$, we deduce that $\operatorname{tr}_k^{(k)} M(t)$ increases on $(0, \infty)$. ∎

COROLLARY 2.5. *Let the assumptions of Theorem 2.1 hold. Then:*

(i) *$E_p(M(t))$ is increasing on $(0, \infty)$, where $E_p(\cdot)$ is the pth elementary symmetric function on the eigenvalues of $M(t)$.*

(ii) *For any unitarily invariant norm $\|\cdot\|$, (i.e., $\|UFV\| = \|F\|$ for all unitary U, V, and $F \in M_n$), $\|M(t)\|$ is increasing.*

(iii) $\operatorname{tr}_i^{(k)} e^{A+B} \leqslant \operatorname{tr}_i^{(k)}(e^A e^B)$.

Proof. (i): $E_p(M(t)) = \operatorname{tr}^{(p)} M(t)$.

(ii): Since $M(t)$ is a Hermitian positive definite matrix, its eigenvalues and singular values are the same. By Theorem 2.1, $\|M(t)\|$ is increasing for all Ky Fan k-norms and so for all unitarity invariant norms.

(iii): For $0 < t < 1$, $\operatorname{tr}_i^{(k)} M(t) \leqslant \operatorname{tr}_i^{(k)} M(1) = \operatorname{tr}_i^{(k)}(e^A e^B)$. Then the result follows from the Lie product formula. ∎

3. THE EQUALITY CASE

In this section we characterize the situation when $\operatorname{tr}_i^{(k)} M(t)$ is not strictly increasing.

THEOREM 3.1. *Let the assumptions of Theorem 2.1 hold. Assume that $\operatorname{tr}_i M(t)$ is constant on a nontrivial interval $[t_1, t_2]$. Then there exists a unitary matrix U such that*

$$U^*AU = \operatorname{Diag}(a_1, \ldots, a_i) \oplus A_1, \qquad U^*BU = \operatorname{Diag}(b_1, \ldots, b_i) \oplus B_1. \qquad (3.2)$$

Furthermore, the first i eigenvalues of $M(t)$ on the interval $(0, t_2]$ are $e^{a_j+b_j}$, $j = 1, \ldots, i$. In particular, on $(0, t_2]$,

$$\operatorname{tr}_i M(t) = \sum_{j=1}^{i} e^{a_j+b_j}. \qquad (3.3)$$

Proof. Let $\omega_1(t) \geqslant \cdots \geqslant \omega_n(t) > 0$ and $\lambda_1(t) \geqslant \cdots \geqslant \lambda_n(t) > 0$ be the eigenvalues of $M(t)$ and $M(t)^t$ respectively. We prove the theorem by the induction on i. Assume that $i = 1$. Theorem 7 in [1] yields that if $\log \lambda_1(t_1) = \log \lambda_1(t_2)$, $0 < t_1 < t_2$, then there exists a unit vector x such that $Ax = ax$, $Bx = bx$, and $\log \lambda_1(t) = (a+b)t$, $t_1 \leqslant t \leqslant t_2$. Next note that e^{a+b} is always an eigenvalue of $M(t)$. Thus, $\omega_1(t) \geqslant e^{a+b} = \omega_1(t_2)$. As $\omega_1(t) = \operatorname{tr}_1 M(t)$ is an increasing function, we deduce that $\omega_1(t) = e^{a+b}$, $0 < t \leqslant t_2$, as we claimed.

Suppose that $1 < i \leqslant n$, and assume that the theorem holds for $i - 1$. Assume that $\operatorname{tr}_i M(t)$ is constant on $[t_1, t_2]$. From the proof of Theorem 2.1 it follows that the equality $\operatorname{tr}_i M(t_1) = \operatorname{tr}_i M(t_2)$ implies $\omega_1(t_1) = \omega_1(t_2), \dots, \omega_i(t_1) = \omega_i(t_2)$. Hence, $\operatorname{tr}_{i-1} M(t_1) = \operatorname{tr}_{i-1} M(t_2)$. By the induction assumption $\operatorname{tr}_{i-1} M(t)$ is constant on $(0, t_2]$. Furthermore, there exists a unitary matrix V such that

$$V^*AV = \operatorname{Diag}(a_1, \dots, a_{i-1}) \oplus C,$$

$$V^*BV = \operatorname{Diag}(b_1, \dots, b_{i-1}) \oplus D,$$

$$\operatorname{tr}_{i-1} M(t) = \sum_{j=1}^{i-1} e^{a_j + b_j}.$$

Hence $\omega_i(t) = \rho((e^{Dt/2} e^{Ct} e^{Dt/2})^{1/t})$ on $[t_1, t_2]$. Apply our argument for $i = 1$ to Hermitian matrices C and D, to deduce the existence of a unitary matrix W such that $W^*CW = a_i \oplus A_1$, $W^*DW = b_i \oplus B_1$, and $\omega_i(t) = e^{a_i + b_i}$ on $(0, t_2]$. Let U be the unitary matrix $V(I_{i-1} \oplus W)$, and our theorem follows. ∎

COROLLARY 3.4. *Let the assumptions of Theorem* 2.1 *hold. Then the following are equivalent*:

(i) *$\operatorname{tr} M(t)$ is not strictly increasing.*
(ii) $\operatorname{tr} M(t)$ *is constant.*
(iii) $\operatorname{tr} e^{A+B} = \operatorname{tr}(e^A e^B)$.
(iv) $AB = BA$.

COROLLARY 3.5. *Let the assumptions of Theorem* 2.1 *hold. Then* $\operatorname{tr}_i^{(k)}\{M(t)$ *is not strictly increasing iff there exists a unitary matrix U such that* $U^*D_k(A)U = \operatorname{Diag}(a_1, \dots, a_i)\ A_1$, $U^*D_k(B)U = \operatorname{Diag}(b_1, \dots, b_i) \oplus B_1$, *and* $\rho(e^{A_1 t_0} e^{B_1 t_0})^{1/t_0} \leqslant \min_{1 \leqslant r \leqslant i} e^{a_r + b_r}$ *for some* $t_0 > 0$. *In particular,* $\operatorname{tr}_i^{(k)} e^{A+B} = \operatorname{tr}_i^{(k)}(e^A e^B)$ *iff there exists a unitary matrix U such that* $U^*D_k(A)U = \operatorname{Diag}(a_1, \dots, a_i) \oplus A_1$, $U^*D_k(B)U = \operatorname{Diag}(b_1, \dots, b_i) \oplus B_1$, *and* $\rho(e^{A_1} e^{B_1}) \leqslant \min_{1 \leqslant r \leqslant i} e^{a_r + b_r}$.

The disadvantage of Corollary 3.5 is that its conditions are stated in terms of the additive adjoints $D_k(A)$, $D_k(B)$ rather than in terms of the original matrices A, B. It would be desirable to restate the results of Corollary 3.5 in terms of A, B. (It seems that the conditions involve common invariant subspaces of A and B.) As an example we claim:

THEOREM 3.6. *Let the assumptions of Theorem* 2.1 *hold. Assume that* $1 < k < n$. *Then the following are equivalent*:

(i) $tr^{(k)} M(t)$ *is not strictly increasing*.
(ii) $\operatorname{tr}^{(k)} M(t)$ *is constant*.
(iii) $\operatorname{tr}^{(k)} e^{A+B} = \operatorname{tr}^{(k)}(e^A e^B)$.
(iv) $AB = BA$.

The proof of this theorem follows from Corollary 3.5 and the following lemma:

LEMMA 3.7. *Let* $F, G \in M_n$ *be diagonable matrices with positive eigenvalues. Assume that* $1 < k < n$. *Then* $C_k(F) = C_k(G)$ *iff* $F = G$. *In particular, for any* $A, B \in H_n$ *one has* $C_k(e^A e^B) = C_k(e^B e^A)$ *iff* $AB = BA$.

Proof. Assume that F is a diagonable matrix with the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$. W.l.o.g. we may assume that $F = \operatorname{Diag}(\lambda_1, \dots, \lambda_n)$. Let $x_i = (\delta_{i1}, \dots, \delta_{in})^T$, $i = 1, \dots, n$, be the standard basis in $\mathbf{C}^n$. Thus $x_{i_1} \wedge \cdots \wedge x_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, is an eigenvector of $C_k(F) = C_k(G)$. That is,

$$C_k(G) x_{i_1} \wedge \cdots \wedge x_{i_k} = Gx_{i_1} \wedge \cdots \wedge Gx_{i_k} = \lambda_{i_1} \cdots \lambda_{i_k} x_{i_1} \wedge \cdots \wedge x_{i_k},$$
$$1 \leqslant i_1 < \cdots < i_k \leqslant n. \tag{3.8}$$

As all the eigenvalues of F are positive, the right-hand side of (32.8) is not equal to zero. Recall that to each k-dimensional subspace of $W \subset \mathbf{C}^n$ there corresponds a point in the Grassmanian $\mathcal{G}_{n,k}$ which is determined by the one-dimensional linear subspace spanned by $y_1 \wedge \cdots \wedge y_k$, where $y_1, \dots, y_k$ is any basis in W. Comparing the last two terms of the above equality, we deduce that $G \operatorname{span}\{x_{i_1}, \dots, x_{i_k}\} = \operatorname{span}\{x_{i_1}, \dots, x_{i_k}\}$. Intersect the above invariant subspaces to deduce that every one-dimensional subspace $\operatorname{span}\{x_i\}$ is an invariant subspace of G. Thus, $Gx_i = \mu_i x_i$, $i = 1, \dots, n$, for some $\mu_i > 0$. The assumption that $C_k(F) = C_k(G)$ means that

$$\sum_{j \in S} \log \lambda_j = \sum_{j \in S} \log \mu_j, \qquad S \subset \{1, \dots, n\}, \quad \operatorname{card} S = k. \tag{3.9}$$

It is left to show that the above conditions for all such S imply that $\lambda_i = \mu_i$, $i = 1, \dots, n$. Summing all equalities in (3.9), we deduce

$$\sum_{j=1}^{n} \log \lambda_j = \sum_{j=1}^{n} \log \mu_j .$$

For fixed $1 \leqslant i \leqslant n$, by summing all equalities in (3.9) with S containing i, we obtain

$$\binom{n-2}{k-2} \sum_{j=1}^{n} \log \lambda_j + \left[\binom{n-1}{k-1} - \binom{n-2}{k-2} \right] \log \lambda_i$$

$$= \binom{n-2}{k-2} \sum_{j=1}^{n} \log \mu_j + \left[\binom{n-1}{k-1} - \binom{n-2}{k-2} \right] \log \mu_i .$$

Hence we deduce that $\log \lambda_i = \log \mu_i$ and so $\lambda_i = \mu_i$.

Let $A, B \in H_n$, $F = e^A e^B$, $G = e^B e^A$. Thus F, G are two diagonable matrices with positive eigenvalues. According to our results, $C_k(e^A e^B) = C_k(e^B e^A)$ implies $e^A e^B = e^B e^A$. That is e^A, e^B have a joint system of orthonormal systems of eigenvectors. Hence A, B have the same joint system of orthonormal vectors, i.e., $AB = BA$. ∎

We conclude this section with an explicit example of $A, B \in H_n$, $n > 2$, with $\operatorname{tr}_1 M(t)$ constant exactly on the interval $(0, \tau)$. Assume that $C, D \in H_{n-1}$ and C, D do not have a common eigenvector. Set $\hat{M}(t) = (e^{tD/2} e^{tC} e^{tD/2})^{1/t}$. Then Theorem 3.1 implies that $\operatorname{tr}_1 \hat{M}(t)$ strictly increases on $(0, \infty)$. For $\tau > 0$ let $a + b = \log \operatorname{tr}_1 \hat{M}(\tau)$. Set $A = \operatorname{Diag}(a) \oplus C$, $B = \operatorname{Diag}(b) \oplus D$. It now follows that

$$\operatorname{tr}_1 M(t) = \operatorname{tr}_1 \hat{M}(\tau), \quad t \in (0, \tau], \qquad \operatorname{tr}_1 M(t) = \operatorname{tr}_1 \hat{M}(t), \quad t \in (\tau, \infty).$$

4. LIMIT MATRICES

THEOREM 4.1. *Let $0 \neq A_1, \dots, A_m \in H_n$, $r_1 > \cdots > r_m \in \mathbf{R}$. Assume that*

$$A(t) = \sum_{i=1}^{m} e^{r_i t} A_i \geqslant 0 \qquad \forall t \geqslant t_0 > 1. \tag{4.2}$$

Then $\lim_{t\to\infty} A(t)^{1/t} = A \geqslant 0$. *Moreover, the maximal eigenvalue of* A *is* e^{r_1}, *and its multiplicity is* rank A_1. *Furthermore, the orthonormal set of eigenvectors corresponding to* e^{r_1} *are* rank A_1 *eigenvectors of* A_1 *corresponding to its positive eigenvalues.*

Proof. In what follows we shall assume that $t > t_0$, and no ambiguity will arise. Set $0 \leqslant L(t) = [A(t)]^{1/t}$. Let $K = \max_{1\leqslant i\leqslant m} \rho(A_i)$. It then follows that $A(t) \leqslant e^{r_1 t} mKI$. Hence, $L(t) \leqslant e^{r_1}(mK)^{1/t} I$. Thus, $L(t)$ is uniformly bounded on $[t_0, \infty)$. Let $\mathscr{A}$ be the set of limit points of $L(t)$ as $t \to \infty$. We need to show that $\mathscr{A}$ consists of one point A. The above inequality shows that $\mathscr{A} \leqslant e^{r_1} I$, i.e. $A \leqslant e^{r_1} I$, $A \in \mathscr{A}$. As $A_1 = \lim_{t\to\infty} e^{-r_1 t} A(t)$, we deduce that $A_1 \geqslant 0$. The assumption that $A_1 \neq 0$ implies that $P_+(A_1)$, the projection of A_1 on its positive eigenvalues, has rank equal to $l_1 = \operatorname{rank} A_1 \geqslant 1$. Let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{l_1} > 0$ be the l_1 positive eigenvalues of A_1. Thus, the l_1 largest eigenvalues of $e^{-r_1 t}A(t)$ are of the form $\alpha_i + \epsilon_i(t)$, $\lim_{t\to\infty} \epsilon_i(t) = 0$, $i = 1, \dots, l_1$. The corresponding l_1 eigenvectors of $A(t)$ converge to the l_1 eigenvectors of A_1 corresponding to its positive eigenvalues. See for example [4]. It then follows that $e^{r_1} P_+(A_1) \leqslant \mathscr{A}$, i.e. $e^{r_1} P_+(A_1) \leqslant A$, $A \in \mathscr{A}$. We thus have shown

$$e^{r_1} P_+(A_1) \leqslant A \leqslant e^{r_1} I, \qquad A \in \mathscr{A}. \tag{4.3}$$

Hence, e^{r_1} is the maximal eigenvalue for every $A \in \mathscr{A}$ of multiplicity l_1 at least. In particular, if $l_1 = n$ we deduce from (4.3) that $\mathscr{A} = \{e^{r_1} I\}$ and we proved the theorem in this case. We now assume that $1 \leqslant l_1 < n$. Observe next that for $1 < k \leqslant n$ either $C_k(A(t)) \equiv 0$ or

$$C_k(A(t)) = A_k(t) = \sum_{i=1}^{m_k} e^{r_{k,i} t} A_{k,i} \geqslant 0, \qquad t \geqslant t_0,$$

$$A_{k,i} \in H_{\binom{n}{k}}, \quad i = 1, \dots, m_k, \quad A_{k,1} \neq 0, \qquad r_{k,1} > \cdots > r_{k,m_k},$$

$$r_{k,1} \leqslant k r_1. \tag{4.4}$$

Furthermore, each $r_{k,i}$ is equal to $r_{j_1} + \cdots + r_{j_k}$, $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant m$. As $C_k(A_1) \neq 0$ for $1 \leqslant k \leqslant l_1$ and $C_k(A_1) = 0$ for $k > l_1$, we deduce that

$$r_{k,1} = k r_1, \quad k = 1, \dots, l_1, \qquad r_{k,1} < k r_1, \quad k > l_1. \tag{4.5}$$

In particular, $A_k(t) \neq 0$ for $1 \leqslant k \leqslant l_1$. Let $L_k(t) = A_k(t)^{1/t}$. As in Section 2, we observe that $L_k(t) = C_k(L(t))$. From the above arguments we deduce that the limit set of $L_k(t)$ when $t \to \infty$ is the set $\mathscr{A}_k = \{C_k(A),\ A \in \mathscr{A}\}$. Let $k_1 = l_1 + 1$. If $A_1(t) = L_{k_1}(t) \equiv 0$, we get that $\mathscr{A}_k = \{0\}$. That is, $\mathscr{A} = \{e^{r_1} P_+(A_1)\}$, and the theorem is established in this case.

Assume that we have (4.4) for $k = k_1$. We now apply (4.3) to the matrix function $L_{k_1}(t)$ to deduce that

$$e^{r_{k_1,1}} P_+\left(A_{k_1,1}\right) \leqslant C_k(A) \leqslant e^{r_{k_1,1}} I, \qquad A \in \mathscr{A}. \tag{4.6}$$

As $r_{k_1,1} < k_1 r_1$, we deduce that e^{r_1} is an eigenvalue of multiplicity l_1 exactly for any $A \in \mathscr{A}$. Apply this argument to the matrix family $A_{k_1}(t)$ to deduce that $e^{r_{k_1,1}}$ is an eigenvalue of multiplicity $l_2 = \operatorname{rank} A_{k_1,1}$ of any $C_{k_1}(A)$, $A \in \mathscr{A}$. This is equivalent to the claim that the $l_1 + 1, \dots, l_1 + l_2$ eigenvalues of any $A \in \mathscr{A}$ are equal to $e^{r_{k_1,1} - l_1 r_1} < e^{r_1}$. Furthermore, the eigenspace of each $C_{k_1}(A)$, $A \in \mathscr{A}$, corresponding to $e^{r_{k_1,1}}$ is the fixed eigenspace spanned by the eigenvectors of $A_{k_1,1}$ corresponding to its positive eigenvalues. This is equivalent to the claim that the l_2-dimensional subspace corresponding to the eigenvalue $e^{r_{k_1,1} - l_1 r_1}$ corresponding to any $A \in \mathscr{A}$ is fixed, i.e. does not depend on A. If $l_1 + l_2 = n$, the theorem is proved. Otherwise, set $k_2 = k_1 + l_2$ and continue as before. The above process stops, which proves that $\mathscr{A}$ consists of exactly one point. ∎

Let $A, B_1, \dots, B_p \in H_n$. Set

$$C_0(t) = e^{tA}, \qquad C_j(t) = e^{tB_j/2} C_{j-1} e^{tB_j/2}, \quad j = 1, \dots, p.$$

Obviously, each C_j is a strictly positive definite hermitian matrix. As $C_0(t)$ and e^{tB_j} have the expansions (4.2), it follows that each $C_j(t)$, $j = 1, \dots, p$, has the expansion (4.2). Finally, note that

$$\det\left[C_p(t)^{1/t}\right] = e^{\operatorname{tr} A + \Sigma_1^p \operatorname{tr} B_j}.$$

Theorem 4.1 yields

COROLLARY 4.7. *Let* $A, B_1, \dots, B_p \in H_n$. *Then* $\lim_{t\to\infty}(e^{tB_p/2} \cdots e^{tB_1/2} e^{tA} e^{tB_1/2} \cdots e^{tB_p/2})^{1/t}$ *exists and is a strictly positive definite matrix.*

Let $L(t)$ be the matrix given in Corollary 4.7. For $p = 1$ we showed that $\operatorname{tr}_i^{(k)} L(t)$ is an increasing function on $(0, \infty)$. It is of interest to know if this is the case for $p > 1$. Alternatively, are there nontrivial examples (i.e., $B_1, \dots, B_p$ do not pairwise commute) such that $\operatorname{tr}_i^{(k)} L(t)$ increases on $(0, \infty)$?

Let $A, B \in H_n$. We now give the explicit formulas to compute the eigenvalues of the limit matrix $C = \lim_{t \to \infty} M(t)$. Set

$$\begin{aligned} A &= U \operatorname{Diag}(\alpha_1, \dots, \alpha_n) U^*, \qquad B = V \operatorname{Diag}(\beta_1, \dots, \beta_n) V^*, \\ U &= (u_1, \dots, u_n), \qquad V = (v_1, \dots, v_n), \qquad U^*U = V^*V = I, \\ e^{tA} &= U \operatorname{Diag}(e^{\alpha_1 t}, \dots, e^{\alpha_n t}) U^*, \\ e^{tB/2} &= V \operatorname{Diag}(e^{\beta_1 t/2}, \dots, e^{\beta_n t/2}) V^*. \end{aligned} \tag{4.8}$$

Let

$$\begin{aligned} d_k = \max \Bigg\{ \sum_{r=1}^{k} \alpha_{i_r} + \beta_{j_r} : \det W(i,j) \neq 0, \\ i = (i_1 < \cdots < i_k), j = (j_1 < \cdots < j_k) \Bigg\}, \\ W = U^*V, \qquad 1 \leqslant k \leqslant n, \qquad d_0 = 0. \end{aligned} \tag{4.9}$$

LEMMA 4.10. *Let $A, B \in H_n$ be of the form* (4.8). *Assume that d_1 is defined by* (4.9). *Then*

$$|v_{j_0}^* u_{i_0}|^2 e^{d_1 t} \leqslant \rho(e^{tB/2} e^{tA} e^{tB/2}) \leqslant n e^{d_1 t},$$

where $d_1 = \alpha_{i_0} + \beta_{j_0}$.

Proof. Note that $e^{tA} = \sum_{i=1}^{n} e^{\alpha_i t} u_i u_i^*$. For any vector y,

$$y^* e^{tA} y = \sum_{i=1}^{n} e^{\alpha_i t} y^* u_i u_i^* y = \sum_{i=1}^{n} e^{\alpha_i t} |y^* u_i|^2 \geqslant e^{\alpha_{i_0} t} |y^* u_{i_0}|^2.$$

In particular, if $y = e^{tB/2} v_{j_0} = e^{\beta_{j_0} t/2} v_{j_0}$, then

$$\rho(e^{tB/2} e^{tA} e^{tB/2}) \geqslant v_{j_0}^* e^{tB/2} e^{tA} e^{tB/2} v_{j_0} \geqslant e^{(\alpha_{i_0} + \beta_{j_0})t} |v_{j_0}^* u_{i_0}|^2 = e^{d_1 t} v_{j_0}^* u_{i_0}|^2.$$

For any unit vector x,

$$e^{\alpha_i t}|u_i^* e^{Bt/2}x|^2 = e^{\alpha_i t}\left|\sum_{j=1}^{n} e^{\beta_j t/2}u_i^* v_j v_j^* x\right|^2 \leqslant \left(\sum_{j=1}^{n} e^{(\alpha_i+\beta_j)t/2}|u_i^* v_j|\,|v_j^* x|\right)^2$$

$$\leqslant e^{d_1 t}\left(\sum_{j=1}^{n}|u_i^* v_j|\,|v_j^* x|\right)^2 \leqslant e^{d_1 t}\left(\sum_{j=1}^{n}|u_i^* v_j|^2\right)\left(\sum_{j=1}^{n}|v_j^* x|^2\right)$$

$$\leqslant e^{d_1 t}.$$

Hence

$$x^* e^{tB/2}e^{tA}e^{tB/2}x = \sum_{i=1}^{n} e^{\alpha_i t}|u_i^* e^{tB/2}x|^2 \leqslant ne^{d_1 t}. \qquad \blacksquare$$

THEOREM 4.11. *Let the assumptions of Theorem* 2.1 *hold. Denote by* $\omega_1(t) \geqslant \cdots \geqslant \omega_n(t) > 0$ *the eigenvalues of* $M(t)$. *Assume that* $d_0, \dots, d_n$ *are defined by* (4.9). *Then* $\lim_{t\to\infty}\omega_k(t) = e^{d_k - d_{k-1}}$, $k = 1, \dots, n$.

Proof. From Lemma 4.10, we have

$$|v_{j_0}^* u_{i_0}|^{2/t}e^{d_1} \leqslant \omega_1(t) \leqslant n^{1/t}e^{d_1}.$$

By letting $t \to \infty$, we obtain

$$\lim_{t\to\infty}\omega_1(t) = e^{d_1}.$$

Next we apply this result to the Hermitian matrices $D_k(A)$ and $D_k(B)$ to obtain

$$\lim_{t\to\infty}\omega_1(t)\cdots\omega_k(t) = e^{d_k}.$$

The above equalities for $k = 1, \dots, n$ imply the theorem. $\blacksquare$

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